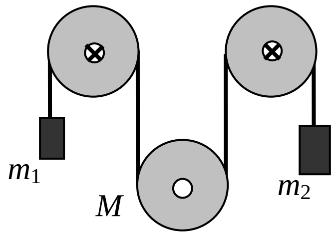
Problem 1: «Three Pulleys».

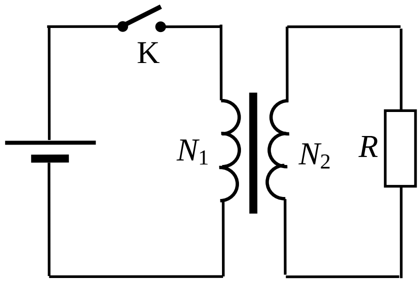


A light inextensible rope is threaded through three identical pulleys. Two pulleys are fixed (cannot rotate) while the third can frictionlessly rotate about an immobile horizontal axis. The pulleys and the rope lie in the same vertical plane. To prevent the rope from sliding off the pulleys the latter have grooves. Coefficient of friction between a pulley groove and the rope is $\mu = \frac{\ln 2}{\pi}$. The mass of the rotating pulley is $M = 4{,}8$ kg. When evaluating its moment of inertia one can neglect the groove and the central hole and consider the pulley to be a uniform disk. The left end of the rope (see the Figure) is loaded with a weight $m_1 = 200$ g and the right end with a weight $m_2 > m_1$. Initially the system is held at rest.

1) Evaluate the ratio of tensions in the sliding rope on both sides of a fixed pulley. Express your answer via the coefficient of friction and calculate its numerical value. (4 points)
2) Determine the critical value of m_2 required for setting the system in motion after it has been released. Express m_2 in terms of the quantities specified in the problem and write down an explicit formula. (3 points)
3) Evaluate the numerical value of m_2 found above. (1 point).
4) Determine an acceleration of m_2 (absolute value) after the system has been released if its mass is greater by the factor $n = 2$ than the critical mass found in 3). Express the answer in terms of the quantities specified in the problem and write down an explicit formula. (4 points)
5) Express the acceleration found in 4) as a fraction of gravitational acceleration g. (1 point)
6) Determine an acceleration of m_2 (absolute value) after the system has been released if its mass is greater by the factor $k = 4$ than the critical mass found in 3). Express the answer in terms of the quantities specified in the problem and write down an explicit formula. (4 points)
7) Express the acceleration found in 6) as a fraction of gravitational acceleration g. (1 point)
8) Under the condition specified in 6) evaluate a tangential acceleration (absolute value) of the rotating pulley circumference right after the system has been released. Express the answer as a fraction of gravitational acceleration g. (4 points)
9) Suppose a value of m_2 exceeds the critical mass found in 3) and now mass M of the rotating pulley can be varied. Determine the values of M for which a type of relative motion of the rope and the rotating pulley remain the same for any m_2. Write down a formula and express it in terms of the quantities specified in the problem. (4 points)
10) Evaluate the ratio of rope tensions on both sides of the rotating pulley if the value of m_2 is specified in 4) while the pulley mass is less by the factor $k = 4$ than the value specified in the problem. The coefficient of friction between the pulley and the rope remains the same. Write down the answer as a simple fraction. (4 points)

TOTAL score of the problem is 30 points.

Problem 2: «Instantaneous switching-on».



A transformer consists of a toroidal ferromagnetic core (magnetic permeability $\mu >> 1$) of a length $l < 1$ m and cross-sectional area S, a primary winding with the number of turns $N_1 = 20$, and a secondary winding with the number of turns $N_2 = 100$. The transformer is a part of the circuit shown in the Figure. The battery emf equals $\mathcal{E} = 24$ V, its internal resistance is $r = 4$ Ohm. The secondary winding is connected to a resistor of $R = 100$ Ohm. Initially there is no current in the windings and the switch K is open. Then the switch is rapidly closed. Its resistance falls to zero very quickly (although not instantly!) during a time much less than $\tau \equiv \frac{\mu_0 \mu N_1^2 S}{lr} = 0{,}05$ sec. Electrical resistance of both windings is much less than 1 Ohm. Dissipation of magnetic flux within the core is negligible.

1) Derive an equation relating a magnetic flux Φ in the core and currents I_1 and I_2 in the primary and secondary windings. Consider a positive direction of current flow in both windings to be the same. (1 point)
2) Consider the circuit including the primary winding and the internal resistance of the battery. Suppose that during switch closing the voltage across the circuit varies as $U_1 = U(t)$. Write down a complete set of first-order differential equations for functions $I_1(t)$ and $I_2(t)$ which describe currents in the primary and secondary windings. (2 points)
3) Specify initial values $I_1(0)$ and $I_2(0)$ (i.e. initial conditions for the set of differential equations derived above). (1 point)
4) Determine a ratio of the current in secondary winding at $t_1 = 2\tau$ to the current at $t_2 = 4\tau$ after the voltage $U(t)$ has been switched on. Evaluate an approximate numerical value of the ratio $\frac{I_2(2\tau)}{I_2(4\tau)}$. (3 points)
5) Determine the maximum value of the current in secondary winding after the switch has been closed. Derive an equation for $I_2^{\max}$ in terms of the quantities specified in the problem and evaluate an approximate numerical value of $I_2^{\max}$. (4 points)
6) Determine a time dependence $I_2(t)$ at $t > \tau$ in terms of the quantities specified in the problem. (3 points)
7) Plot approximately the dependence $I_2(t)$ from the moment the switch starts being closed to $T = 0{,}5$ sec. (3 points)
8) What is the maximum current in the primary winding after the switch has been closed? Derive the equation for $I_1^{\max}$ in terms of the quantities specified in the problem and evaluate the numerical value. (2 points)
9) Derive a time dependence $I_1(t)$ for $t > \tau$ in terms of the quantities specified in the problem. (3 points)
10) Plot approximately the dependence $I_1(t)$ from the moment the switch starts being closed to $T = 0{,}5$ sec. (3 points)

TOTAL score for the problem is 25 points.

Problem 3: «Planet Cronus».

Let us imagine that some civilisation left an artificial planet in the Solar system. The planet orbit lies beyond the orbit of Pluto, so let us call it «Cronus». The Cronus orbit is very close to a circle of radius R_K = 50 au (1 astronomical unit (au) is roughly equal to the average distance from the Earth to the Sun), the planet itself is a sphere of radius $r = 5000$ km made of a solid material with a density $\rho \approx 9$ g/cm^3 and good heat conducting properties, and an average heat capacity $c \approx 0{,}3$ J/(g·K). A Cronus «year» lasts 20 «solar days», the planet rotates around its axis in the same direction as its orbital rotation around the Sun, and the Cronus rotation axis is perpendicular to its orbital plane. (Notice, that a planet «year» is a period of rotation of its centre of mass around the Sun and «solar day» is the average interval between two «noondays», the moments when the Sun is at maximum elevation on planet sky). Cronus has no satellites, it has an atmosphere consisting of nitrogen, helium, neon, and water vapour. At the beginning of observation Cronus is a rather hot place. This is due to a uniform layer of radioactive material under the planet surface which provides the heat. The material half-life is $\tau_{1/2} = 5000$ Cronus «years». So, one day a human-made probe landed on the Cronus surface and measured temperature and relative humidity of atmospheric layer at the surface. According to the measurements $T_0 \approx 330$ K and $\varphi_0 \approx 80$ %. It turns out the temperature within a dense part of Cronus atmosphere (which contains about 99% of its mass) decreases with altitude as $T(h) \approx T_0 \cdot \left(1 - \dfrac{h}{6H}\right)$, where $H \approx 10$ km. This distribution remains almost constant (if disturbed the atmosphere returns to this state in about 1 «year»). The atmosphere is so «pure» that all its water remains in vapour state and there are almost no clouds. Necessary constants:

- The gravitational constant $G \approx 6{,}67 \cdot 10^{-11}$ m^3·sec^{-2}·kg^{-1}.
- Temperature of the Sun photosphere $T_1 \approx 6000$ K, solar radius $r_1 \approx 0{,}00465$ au, the Stephan-Boltzmann constant $\sigma \approx 5{,}67 \cdot 10^{-8}$ W/(m^2·K^4).
- The specific heat of evaporation of water can be considered to be almost temperature independent and approximately equal to $\kappa \approx 2366$ J/g at T_0. The molar mass of water $\mu \approx 18$ g/mole and the universal gas constant $R \approx 8{,}31$ J/(mole·K).
- The boiling point of water under the normal atmospheric pressure of 101,3 kPa equals 373 K.

Using the above information you should be able to answer the following questions (indicate appropriate units of measurement at all numerical results).

1) Determine how many Earth years does a Cronus «year» have. (1 point)
2) Evaluate angular velocity of the planet rotation around its axis relative to the frame of reference of «distant stars». Evaluate a ratio of the planet centripetal acceleration to the free fall acceleration at the planet equator. The answer must consist of two numbers calculated with an accuracy of 1% at least. (2 points)
3) Determine the power of solar radiation absorbed by Cronus by assuming that the planet absorbs all incident radiation. Express the answer in terms of the quantities specified in the problem. (1 point)
4) Evaluate a numerical ratio of the absorbed power to the power radiated by Cronus at the beginning of observations. (2 points)
5) Evaluate a time for which the surface temperature of Cronus would decrease by 4 K if the radioactive «heating» abruptly vanished. Write down the formula in terms of the quantities specified in the problem and numerical value in Cronus years. (3 points)

6) Use the above estimates to develop a mathematical model of the planet surface cooling providing the radioactive heating operates all the time and the surface cools naturally. Determine an approximate time dependence of surface temperature T. Write down the formula in terms of the quantities specified in the problem. (3 points)

7) Evaluate a time for which the surface temperature of Cronus would decrease by 4 K (with radioactive heating present). Write down the formula in terms of the quantities specified in the problem. Obtain a numerical value expressed in Cronus years. (2 points)

Further analysis relies on properties of water vapour. Consider two isotherms of water vapour on a pV-diagram at temperatures T and $T + dT$. Let the isotherms correspond to a slow transition from vapour to water and back (it is well known that vapour pressure does not change along such a transition). By «closing» these isotherms near their edges with two small adiabatics you should be able to calculate the following quantities: a) an efficiency of the obtained cycle; b) a work done during the cycle; and c) a heat taken from a «heater» by vapour during the cycle.

8) Using a relation between the above quantities (a-c) and discarding a volume of the liquid phase compared to a volume of vapour of the same mass determine a slope of the temperature dependence of the saturated water vapour pressure at a given point of T and p_v. The answer is a formula for $\dfrac{dp_\mathrm{v}}{dT}$ as a function of T and p_v. (3 points)

9) Under the same assumptions and considering the heat of evaporation of water to be almost independent of temperature determine a temperature dependence of pressure p_v of saturated water vapour near T_0. Write down the formula. (4 points)

10) Using the obtained results determine the planet surface temperature T_1 at which water vapour begin to condense. Write down the formula expressed in terms of the quantities specified in the problem. Evaluate the numerical value. A loss of water «outside» the atmosphere is negligible. (5 points)

11) Determine the time after beginning of observation when water vapour starts to condense. Write down the formula in terms of the quantities specified in the problem and evaluate the numerical value expressed in Cronus years. (5 points)

12) Evaluate the maximum depth of ocean on the Cronus surface built up of water vapour condensed from the atmosphere. Write down the formula in terms of the quantities specified in the problem, the pressure $p_\mathrm{v}(T_0)$ of saturated water vapour at T_0, and a density of liquid water ρ_l. (2 points)

13) Determine (under the same assumptions) a relation between the surface temperature and a ratio of ocean depth to the maximum possible depth at $T < T_1$ but before the complete condensation took place (i.e. when ocean depth is not close to the maximum value). Write down the equation in terms of the quantities specified in the problem. (6 points)

14) Determine the time when the average ocean depth on the Cronus surface becomes a quarter of the maximum. Write down the formula in terms of the quantities specified in the problem and evaluate its numerical value in Cronus years. (4 points)

15) Is it necessary to take into account the heat of vaporisation when evaluating the time of planet cooling at $T < T_1$ if the accuracy of calculation is 5%? And if the accuracy is 0,5%? Answer to both questions «yes» or «no». (2 points)

TOTAL score for the problem is 45 points.